

Structure of a class of Lie algebras of Block type ¹

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Abstract. Let $\mathcal{B}(q)$ be a class of Lie algebras of Block type with basis $\{L_{\alpha,i} \mid \alpha, i \in \mathbb{Z}, i \geq 0\}$ and relations $[L_{\alpha,i}, L_{\beta,j}] = (\beta(i+q) - \alpha(j+q))L_{\alpha+\beta, i+j}$, where q is a positive integer. In this paper, it is shown that $\mathcal{B}(q)$ are different from each other for distinct positive integers q 's. The automorphism groups, the derivation algebras and the central extensions of all $\mathcal{B}(q)$ are also uniformly and explicitly described, which generalize some previous results.

Key words: Lie algebras of Block type; automorphism; derivation; central extension.

Mathematics Subject Classification (2000): 17B05; 17B40; 17B56; 17B65; 17B68.

1. Introduction

In [1], Block introduced a class of infinite dimensional simple Lie algebras. Partially due to their relations to the (centerless) Virasoro algebra, generalizations of Lie algebras of this type (usually referred to as *Lie algebras of Block type*) have received many authors' interests (see, e.g., [3, 8–13, 16, 19, 20, 22, 23]). These Lie algebras are constructed from pairs $(\mathcal{A}, \mathcal{D})$ consisting of a commutative associative algebra \mathcal{A} with an identity element and a finite dimensional abelian derivation subalgebra \mathcal{D} such that \mathcal{A} is \mathcal{D} -simple (such pairs are classified in [14]). The representation theory for the simple Lie algebras of Block type is far from being well developed, except for some quasifinite representations (see, for example, [10–12, 17]), which is partially because the structure theory for the Lie algebras of Block type is not well developed yet.

In the present paper, we concentrate on a class of Lie algebras $\mathcal{B}(q)$, where q is a positive integer, with basis $\{L_{\alpha,i} \mid \alpha \in \mathbb{Z}, i \in \mathbb{Z}_+\}$ and brackets

$$[L_{\alpha,i}, L_{\beta,j}] = (\beta(i+q) - \alpha(j+q))L_{\alpha+\beta, i+j} \quad \text{for } \alpha, \beta \in \mathbb{Z}, i, j \in \mathbb{Z}_+. \quad (1.1)$$

The structure theory, including automorphism group, derivation algebra and central extension, for the special case $\mathcal{B}(1)$ were considered in [18, 21]. Our goal is to study the structure theory (and later the representation theory) for the whole class of Lie algebras $\mathcal{B}(q)$ not only for a particular Lie algebra $\mathcal{B}(1)$. Our first result is the following.

¹Supported by NSF grant 10825101 of China

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Theorem 1.1 (Isomorphism Theorem) *Lie algebras $\mathcal{B}(q)$ are different from each other for distinct positive integers q 's, namely,*

$$\mathcal{B}(q_1) \cong \mathcal{B}(q_2) \iff q_1 = q_2.$$

The quasifinite representations of Lie algebras of Block type were initially studied in [10, 11], which seem to be intrigued by Mathieu's classification of Harish-Chandra modules over the well-known Virasoro algebra in [7]. As pointed in [17], the central extensions of $\mathcal{B}(1)$, which can be realized as a subalgebra of the Block type Lie algebra \mathcal{B} studied in [10] and which is quite different from \mathcal{B} , contains a subalgebra isomorphic to the Virasoro algebra, thus the representations for $\mathcal{B}(1)$ and its central extensions will be much more interesting. We will show in Theorem 4.1 that the central extension, denoted $\widehat{\mathcal{B}(q)}$, of $\mathcal{B}(q)$ for $q \geq 1$ is given by

$$[L_{\alpha,i}, L_{\beta,j}] = (\beta(i+q) - \alpha(j+q)) L_{\alpha+\beta, i+j} + \delta_{\alpha+\beta, 0} \delta_{i,0} \delta_{j,0} \frac{\alpha^3 - \alpha}{12} c,$$

for $\alpha, \beta \in \mathbb{Z}$, $i, j \in \mathbb{Z}_+$. One sees that $\widehat{\mathcal{B}(q)}$ contains a subalgebra with basis $\{q^{-1}L_{\alpha,0}, c \mid \alpha \in \mathbb{Z}\}$ isomorphic to the Virasoro algebra. One also sees that $\mathcal{B}(q)$ contains a subalgebra with basis $\{q^{-1}L_{\alpha, qi} \mid \alpha \in \mathbb{Z}, i \in \mathbb{Z}_+\}$, which is isomorphic to $\mathcal{B}(1)$. Due to these two facts and the above isomorphism theorem, one may expect that the representation theory of $\widehat{\mathcal{B}(q)}$ ($q \geq 2$) may be more interesting than that of $\widehat{\mathcal{B}(1)}$. We would also like to point out that although $\mathcal{B}(q)$ is \mathbb{Z} -graded with respect to eigenvalues of $\text{ad}_{L_{0,0}}$, it is not finitely-generated \mathbb{Z} -graded, thus some classical methods due to Farnsteiner [4] (which are efficient for finitely generated graded Lie algebras), cannot be applied in our case here.

Now we outline our main results in the present paper. In Section 2, after giving the proof of Theorem 1.1, we describe the automorphism group of $\mathcal{B}(q)$ (Theorem 2.7), which in particular shows that $\mathcal{B}(q)$ has no nontrivial inner automorphisms. In Section 3, by employing a technique developed in [16], we characterize the structure of the derivation algebra of $\mathcal{B}(q)$ and prove that the first cohomology group of $\mathcal{B}(q)$ with coefficients in its adjoint module is one-dimensional (Theorem 3.1). Finally in Section 4, we determine the central extensions and the second cohomology group of $\mathcal{B}(q)$ and prove that $\widehat{\mathcal{B}(q)}$ is an essentially unique nontrivial one-dimensional central extension of $\mathcal{B}(q)$ (Theorem 4.1).

Throughout this paper, q will denote a fixed positive integer. All the vector spaces are assumed over the complex field \mathbb{C} . As usual, we denote by \mathbb{Z} the ring of integers and by \mathbb{Z}_+ the set of nonnegative integers.

2. Automorphisms of $\mathcal{B}(q)$

An element $F \in \mathcal{B}(q)$ is called

- (1) *ad-locally finite* if for any given $v \in \mathcal{B}(q)$, the subspace $\text{Span}\{\text{ad}_F^m(v) \mid m \in \mathbb{Z}_+\}$ of $\mathcal{B}(q)$ is finite dimensional;
- (2) *ad-locally nilpotent* if for any given $v \in \mathcal{B}(q)$, there exists some $N > 0$ such that $\text{ad}_F^N(v) = 0$.

Denote by $\text{Aut } \mathcal{B}(q)$ the *automorphism group* of $\mathcal{B}(q)$, and $\text{Int } \mathcal{B}(q)$ the *inner automorphism group* of $\mathcal{B}(q)$, i.e., the subgroup of $\text{Aut } \mathcal{B}(q)$ generated by \exp^{ad_x} for ad-locally nilpotent elements x 's. Recall that the centerless Virasoro algebra Vir (or Witt algebra) with basis $\{L_\alpha \mid \alpha \in \mathbb{Z}\}$ is defined by the following commutation relations:

$$[L_\alpha, L_\beta] = (\beta - \alpha)L_{\alpha+\beta} \quad \text{for } \alpha, \beta \in \mathbb{Z}.$$

In this section, we first prove that $\mathcal{B}(q)$ contains a unique locally finite element $L_{0,0}$ (up to scalars), which is not locally nilpotent (thus $\text{Int } \mathcal{B}(q)$ is trivial). Then we introduce a lemma which gives some relations between Vir and $\mathcal{B}(q)$. After that, by giving the structure of automorphism group of Vir , and introducing two useful lemmas, we present a proof of Theorem 1.1. Finally we completely characterize the structure of $\text{Aut } \mathcal{B}(q)$ (Theorem 2.7).

Lemma 2.1 *We have the following facts:*

- (1) *Up to scalars, $L_{0,0}$ is the unique locally finite element of $\mathcal{B}(q)$.*
- (2) *$L_{0,0}$ is not locally nilpotent, thus $\text{Int } \mathcal{B}(q)$ is trivial.*

Proof. (1) Introduce a *total order* \prec on $\mathbb{Z} \times \mathbb{Z}_+$ defined by

$$(\alpha, i) \prec (\beta, j) \quad \text{if } \alpha < \beta \quad \text{or} \quad \alpha = \beta, i > j.$$

Let $F = \sum_{(\alpha, i) \in I_F} \lambda_{\alpha, i} L_{\alpha, i}$ be any locally finite element of $\mathcal{B}(q)$, where I_F is a finite subset of $\mathbb{Z} \times \mathbb{Z}_+$. First assume that there exists $\lambda_{\alpha, i} \neq 0$ for some $\alpha < 0$. Set

$$(\alpha_0, i_0) = \min\{(\alpha, i) \in I_F \mid \lambda_{\alpha, i} \neq 0\} \quad (\text{under the sense of total order } \prec).$$

Then clearly $\alpha_0 < 0$. By rescaling F , we may suppose

$$F = L_{\alpha_0, i_0} + \sum_{(\alpha_0, i_0) \prec (\alpha, i)} \lambda_{\alpha, i} L_{\alpha, i}.$$

In this case we say that F has the *minimal term* L_{α_0, i_0} . Recall that $[L_{\alpha_0, i_0}, L_{\beta, j}] = f(\beta, j)L_{\alpha_0+\beta, i_0+j}$, where $f(\beta, j) := \beta(i_0 + q) - \alpha_0(j + q)$. For any given $j_0 \in \mathbb{Z}_+$, we can choose small enough β_0 such that $f(\beta_0, j_0) < 0$ and then

$$f(\beta_0 + k\alpha_0, j_0 + ki_0) = f(\beta_0, j_0) + k\alpha_0 q < 0 \quad \text{for all } k \in \mathbb{Z}_+,$$

which implies that $\text{ad}_F^k(L_{\beta_0, j_0})$, with minimal terms $L_{\beta_0+k\alpha_0, j_0+ki_0}$, are linear independent for all k , thus F is not ad-locally finite. Hence $\lambda_{\alpha, i} = 0$ for all $\alpha < 0$. Similarly, we can prove that $\lambda_{\alpha, i} = 0$ for all $\alpha > 0$. In turn, we can rewrite $F = \sum_{i \in I'_F} \lambda_{0, i} L_{0, i}$, where I'_F is a finite subset of \mathbb{Z}_+ . If there exists $\lambda_{0, i} \neq 0$ for some $i > 0$, then similarly we can assume

$$F = L_{0, i_0} + \sum_{i_0 > i} \lambda_{0, i} L_{0, i}.$$

Note that $[L_{0, i_0}, L_{\beta, j}] = g(\beta)L_{\beta, i_0+j}$, where $g(\beta) := \beta(i_0 + q)$. In particular, we see that $g(1) = i_0 + q > 0$, which implies that $\text{ad}_F^k(L_{1, j})$ are linear independent for all k . So $\lambda_{0, i} = 0$ for all $i > 0$. Thus $F = \lambda_{0, 0} L_{0, 0}$ for some $\lambda_{0, 0} \in \mathbb{C}$, namely, $L_{0, 0}$ is up to scalars the unique locally finite element of $\mathcal{B}(q)$.

(2) Note that any locally nilpotent element must be locally finite element by definition. Since $\text{ad}_{L_{0, 0}}^N L_{1, 0} = q^N L_{1, 0} \neq 0$ for any positive integer N , we see that $L_{0, 0}$ is not locally nilpotent. By (1), $\mathcal{B}(q)$ does not contain any nonzero locally nilpotent elements (in particular, the inner automorphism group of $\mathcal{B}(q)$ is trivial). \square

Lemma 2.2 *Let $\text{Span}\{\bar{L}_\alpha \mid \alpha \in \mathbb{Z}\}$ be a subalgebra of $\mathcal{B}(q)$, which is isomorphic to Vir , namely $[\bar{L}_\alpha, \bar{L}_\beta] = (\beta - \alpha)\bar{L}_{\alpha+\beta}$. If $\bar{L}_0 \in \mathbb{C}L_{0, 0}$, then there exists some $s_0 \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ such that $\bar{L}_\alpha \in \mathbb{C}L_{s_0\alpha, 0}$ for all $\alpha \in \mathbb{Z}$.*

Proof. Assume $\bar{L}_0 = q^{-1}a_0 L_{0, 0}$ for some $a_0 \in \mathbb{C}^*$. Let $0 \neq \alpha \in \mathbb{Z}$. Write $\bar{L}_\alpha = \sum_{(\beta, j) \in J_\alpha} \mu_{\beta, j}^\alpha L_{\beta, j}$, where J_α is a finite subset of $\mathbb{Z} \times \mathbb{Z}_+$. Then

$$\alpha \sum_{(\beta, j) \in J_\alpha} \mu_{\beta, j}^\alpha L_{\beta, j} = \alpha \bar{L}_\alpha = [\bar{L}_0, \bar{L}_\alpha] = \left[q^{-1}a_0 L_{0, 0}, \sum_{(\beta, j) \in J_\alpha} \mu_{\beta, j}^\alpha L_{\beta, j} \right] = \sum_{(\beta, j) \in J_\alpha} \beta a_0 \mu_{\beta, j}^\alpha L_{\beta, j},$$

which implies

$$(\alpha - \beta a_0) \mu_{\beta, j}^\alpha = 0 \quad \text{for all } 0 \neq \alpha \in \mathbb{Z} \text{ and } (\beta, j) \in J_\alpha. \quad (2.1)$$

Obviously $\mu_{0, j}^\alpha = 0$ for all $0 \neq \alpha \in \mathbb{Z}$ and $(0, j) \in J_\alpha$. Note that for any nonzero α , there exists at least one $\mu_{\beta, j}^\alpha \neq 0$ with $\beta \neq 0$. In particular, take $\alpha = 1$, there exist

$s_0 \in \mathbb{Z}^*$ and some j_0 such that $\mu_{s_0, j_0}^1 \neq 0$, and so $a_0 = \frac{1}{s_0}$. Thus (2.1) implies in general $\mu_{\beta, j}^\alpha = 0$ if $\beta \neq s_0 \alpha$. Hence we can rewrite $\bar{L}_\alpha = \sum_{j \in J'_\alpha} \mu_j^\alpha L_{s_0 \alpha, j}$, where $\mu_j^\alpha = \mu_{s_0 \alpha, j}^\alpha$ and $J'_\alpha = \{j \mid (s_0 \alpha, j) \in J_\alpha\} \subset \mathbb{Z}_+$. Then

$$\begin{aligned} 2\alpha(qs_0)^{-1}L_{0,0} &= [\bar{L}_{-\alpha}, \bar{L}_\alpha] = \left[\sum_{i \in J'_{-\alpha}} \mu_i^{-\alpha} L_{-s_0 \alpha, i}, \sum_{j \in J'_\alpha} \mu_j^\alpha L_{s_0 \alpha, j} \right] \\ &= \sum_{(i,j) \in J'_{-\alpha} \times J'_\alpha} (i+j+2q)\alpha s_0 \mu_i^{-\alpha} \mu_j^\alpha L_{0, i+j}. \end{aligned} \quad (2.2)$$

Let $i_0 = \max\{i \in J'_{-\alpha} \mid \mu_i^{-\alpha} \neq 0\}$, $j_0 = \max\{j \in J'_\alpha \mid \mu_j^\alpha \neq 0\}$. If $i_0 + j_0 > 0$, then the right hand side of (2.2) contains the nonzero term $(i_0 + j_0 + 2q)\alpha s_0 \mu_{i_0}^{-\alpha} \mu_{j_0}^\alpha L_{0, i_0 + j_0}$, which is not in $\mathbb{C}L_{0,0}$ (since $i_0, j_0 \geq 0$). Thus $i_0 = j_0 = 0$, in particular, $\bar{L}_\alpha \in \mathbb{C}L_{s_0 \alpha, 0}$. \square

Proposition 2.3 (See for example [9, 15]) *The following two maps are automorphisms of Vir:*

$$\begin{aligned} \bar{\theta}_\mu : \text{Vir} &\rightarrow \text{Vir}, & L_\alpha &\mapsto \mu^\alpha L_\alpha, \\ \bar{\delta}_s : \text{Vir} &\rightarrow \text{Vir}, & L_\alpha &\mapsto s L_{s\alpha}, \end{aligned}$$

where $\mu \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $s \in \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$. Furthermore, the structure of the automorphism group of Vir, denoted $\text{Aut}(\text{Vir})$, is given by $\text{Aut}(\text{Vir}) \cong \mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$.

Let $\mathcal{B}(q_1)$ and $\mathcal{B}(q_2)$ be two Lie algebras of Block type defined as (1.1). We use the notations $L_{\alpha, i}$ and $L'_{\alpha, i}$ to stand for the base elements of $\mathcal{B}(q_1)$ and $\mathcal{B}(q_2)$ respectively. Suppose $\tau : \mathcal{B}(q_1) \rightarrow \mathcal{B}(q_2)$ is a Lie algebra isomorphism. We want to prove $q_1 = q_2$. First we give the following two useful lemmas.

Lemma 2.4 *We have $\tau(L_{\alpha, 0}) = s q_1 q_2^{-1} \mu^\alpha L'_{s\alpha, 0}$, where $\mu \in \mathbb{C}^*$, $s \in \{\pm 1\}$.*

Proof. Let $L'_\alpha = \tau(q_1^{-1} L_{\alpha, 0})$ for $\alpha \in \mathbb{Z}$. Since $\mathcal{V}_1 = \text{Span}\{q_1^{-1} L_{\alpha, 0} \mid \alpha \in \mathbb{Z}\}$ is a subalgebra of $\mathcal{B}(q_1)$ isomorphic to Vir, we see that $\mathcal{V}_2 = \text{Span}\{L'_\alpha \mid \alpha \in \mathbb{Z}\}$ is a subalgebra of $\mathcal{B}(q_2)$ isomorphic to Vir. Furthermore, since $L_{0,0}$ and $L'_{0,0}$ are up to scalars the unique ad-locally finite elements in $\mathcal{B}(q_1)$ and $\mathcal{B}(q_2)$ respectively, we must have $L'_0 = \tau(q_1^{-1} L_{0,0}) \in \mathbb{C}L'_{0,0}$. Hence Lemma 2.2 implies that there exists $s \in \mathbb{Z}^*$ such that $L'_\alpha \in \mathbb{C}L'_{s\alpha, 0}$. Analogously, there exists $s' \in \mathbb{Z}^*$ such that $\tau^{-1}(L'_{\alpha, 0}) \in \mathbb{C}L_{s'\alpha, 0}$ for all $\alpha \in \mathbb{Z}$. In particular $ss' = 1$. Thus $s = \pm 1$.

We may assume that $s = 1$ as the case for $s = -1$ is similar. Thus we can suppose $\tau(L_{\alpha,0}) = q_1 \mu_\alpha L'_{\alpha,0}$ for some $\mu_\alpha \in \mathbb{C}^*$ for $\alpha \in \mathbb{Z}$. Applying τ to $[L_{1,0}, L_{\alpha,0}] = (\alpha - 1)q_1 L_{\alpha+1,0}$ gives $(\alpha - 1)(\mu_{\alpha+1} - q_2 \mu_1 \mu_\alpha) = 0$, which by induction implies

$$\mu_\alpha = \begin{cases} \mu_2 (q_2 \mu_1)^{\alpha-2} & \text{if } \alpha \geq 2, \\ q_2^{-1} (q_2 \mu_1)^\alpha & \text{if } \alpha \leq 1. \end{cases} \quad (2.3)$$

Applying τ to $[L_{-2,0}, [L_{2,0}, L_{3,0}]] = 7q_1^2 L_{3,0}$, we obtain $\mu_{-2} \mu_2 q_2^2 = 1$, which together with (2.3) gives $\mu_\alpha = q_2^{-1} (q_2 \mu_1)^\alpha$. Hence $\tau(L_{\alpha,0}) = q_1 q_2^{-1} (q_2 \mu_1)^\alpha L'_{\alpha,0}$ for $\alpha \in \mathbb{Z}$. \square

Lemma 2.5 *We have $q_1 | q_2$ and $\tau(L_{0,i}) = \nu_i L'_{0,q_1^{-1}q_2 i}$, where $\nu_i \in \mathbb{C}^*$.*

Proof. Assume that $\tau(L_{0,i}) = \sum_{(\beta,j) \in K_i} \nu_{\beta,j}^i L'_{\beta,j}$, where K_i is some finite subset of $\mathbb{Z} \times \mathbb{Z}_+$. Applying τ to the equation $[L_{0,0}, L_{0,i}] = 0$, we obtain by Lemma 2.4 that

$$\sum_{(\beta,j) \in K_i} s q_1 \beta \nu_{\beta,j}^i L'_{\beta,j} = 0,$$

which implies that $\nu_{\beta,j}^i = 0$ if $\beta \neq 0$. In turn, $\tau(L_{0,i})$ can be rewritten as

$$\tau(L_{0,i}) = \sum_{j \in K'_i} \nu_j^i L'_{0,j}, \quad \text{where } \nu_j^i = \nu_{0,j}^i, \text{ and } K'_i = \{j \mid (0,j) \in K_i\}.$$

Applying τ to $[L_{-1,0}, [L_{1,0}, L_{0,i}]] = -(i + q_1)(i + 2q_1)L_{0,i}$, we obtain by Lemma 2.4 that

$$\sum_{j \in K'_i} (q_1 j - q_2 i)(q_1 j + q_2 i + 3q_1 q_2) \nu_j^i L'_{0,j} = 0.$$

Since $i, j \geq 0$ and $q_1, q_2 > 0$, the above implies that $\nu_j^i = 0$ if $j \neq q_1^{-1} q_2 i$. If $q_1 \nmid q_2$, then we see that $\tau(L_{0,1}) = 0$, which is impossible since τ is an isomorphism. Therefore $q_1 | q_2$ and $\tau(L_{0,i}) = \nu_i L'_{0,q_1^{-1}q_2 i}$, where $\nu_i = \nu_{q_1^{-1}q_2 i}^i \in \mathbb{C}^*$. This completes the proof. \square

Proof of Theorem 1.1. Now we give the proof of Theorem 1.1. Let τ be a Lie algebra isomorphism from $\mathcal{B}(q_1)$ to $\mathcal{B}(q_2)$. By Lemma 2.5, we have $q_1 | q_2$. On the other hand, τ^{-1} is also an Lie algebra isomorphism from $\mathcal{B}(q_2)$ to $\mathcal{B}(q_1)$, then $q_2 | q_1$. Since $q_1, q_2 > 0$, we have $q_1 = q_2$. \square

Corollary 2.6 *Let $\sigma \in \text{Aut } \mathcal{B}(q)$. There exist $\mu \in \mathbb{C}^*$, $s \in \{\pm 1\}$ and $\nu_i \in \mathbb{C}^*$ for $i \in \mathbb{Z}_+$, such that*

- (1) $\sigma(L_{\alpha,0}) = s\mu^\alpha L_{s\alpha,0}$,
- (2) $\sigma(L_{0,i}) = \nu_i L_{0,i}$.

Proof. Parts (1) and (2) follow directly from Lemma 2.4 and 2.5 respectively (one will see a further description of $\sigma(L_{0,i})$ in the following theorem). \square

Motivated by the Proposition 2.3 and Corollary 2.6, for any $\mu, \nu \in \mathbb{C}^*$, $s \in \{\pm 1\}$, one can define the following three kinds of linear transformations of $\mathcal{B}(q)$:

$$\begin{aligned}\theta_\mu : \mathcal{B}(q) &\rightarrow \mathcal{B}(q) & L_{\alpha,i} &\mapsto \mu^\alpha L_{\alpha,i}; \\ \eta_\nu : \mathcal{B}(q) &\rightarrow \mathcal{B}(q) & L_{\alpha,i} &\mapsto \nu^i L_{\alpha,i}; \\ \delta_s : \mathcal{B}(q) &\rightarrow \mathcal{B}(q) & L_{\alpha,i} &\mapsto sL_{s\alpha,i}.\end{aligned}$$

One can easily check that they are all automorphisms of $\mathcal{B}(q)$. Furthermore, we have the following facts.

- $\{\theta_\mu \mid \mu \in \mathbb{C}^*\} \cong \mathbb{C}^*$ is a subgroup of $\text{Aut}\mathcal{B}(q)$, where $\theta_{\mu_1}\theta_{\mu_2} = \theta_{\mu_1\mu_2}$ for $\mu_1, \mu_2 \in \mathbb{C}^*$.
- $\{\eta_\nu \mid \nu \in \mathbb{C}^*\} \cong \mathbb{C}^*$ is a subgroup of $\text{Aut}\mathcal{B}(q)$, where $\eta_{\nu_1}\eta_{\nu_2} = \eta_{\nu_1\nu_2}$ for $\nu_1, \nu_2 \in \mathbb{C}^*$.
- $\{\delta_s \mid s = \pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$ is a subgroup of $\text{Aut}\mathcal{B}(q)$.

Theorem 2.7 *Let $\sigma \in \text{Aut}\mathcal{B}(q)$, we have $\sigma(L_{\alpha,i}) = s\mu^\alpha\nu^i L_{s\alpha,i}$ for some $\mu, \nu \in \mathbb{C}^*$, $s \in \{\pm 1\}$. In particular, $\text{Aut}\mathcal{B}(q) \cong (\mathbb{C}^* \times \mathbb{C}^*) \rtimes \mathbb{Z}/2\mathbb{Z}$.*

Proof. Let $\sigma \in \text{Aut}\mathcal{B}(q)$, by Corollary 2.6, we have $\sigma(L_{\alpha,0}) = s\mu^\alpha L_{s\alpha,0}$ and $\sigma(L_{0,i}) = \nu_i L_{0,i}$ for some $\mu, \nu_i \in \mathbb{C}^*$, $s \in \{\pm 1\}$. Applying σ to the equation $[L_{\alpha,0}, L_{0,i}] = -\alpha(i+q)L_{\alpha,i}$ gives

$$\sigma(L_{\alpha,i}) = \mu^\alpha \nu_i L_{s\alpha,i} \quad \text{for } \alpha \in \mathbb{Z}, i \in \mathbb{Z}_+.$$

Applying σ to $[L_{0,1}, L_{1,i}] = (1+q)L_{1,i+1}$, we obtain $\nu_{i+1} = s\nu_1\nu_i$, which implies that $\nu_i = s\nu^i$ for some $\nu \in \mathbb{C}^*$. In turn, $\sigma(L_{\alpha,i}) = s\mu^\alpha\nu^i L_{s\alpha,i}$. In particular, based on facts of the three maps θ_μ, η_ν and δ_s stated above, we see that $\text{Aut}\mathcal{B}(q) \cong (\mathbb{C}^* \times \mathbb{C}^*) \rtimes \mathbb{Z}/2\mathbb{Z}$. \square

3. Derivations of $\mathcal{B}(q)$

Note that $\mathcal{B}(q) = \bigoplus_{\alpha \in \mathbb{Z}} \mathcal{B}(q)_\alpha$ is a \mathbb{Z} -graded Lie algebra, where $\mathcal{B}(q)_\alpha = \text{span}\{L_{\alpha,i} \mid i \in \mathbb{Z}_+\}$. Recall that a *derivation* d of $\mathcal{B}(q)$ is a linear transformation on $\mathcal{B}(q)$ such that

$$d([x, y]) = [d(x), y] + [x, d(y)] \quad \text{for } x, y \in \mathcal{B}(q).$$

Denote $\text{Der } \mathcal{B}(q)$ and $\text{ad } \mathcal{B}(q)$ the space of the derivations and *inner derivations* of $\mathcal{B}(q)$, respectively. Elements in $\text{Der } \mathcal{B}(q) \setminus \text{ad } \mathcal{B}(q)$ are called *outer derivations*. We say that a derivation d has *degree* α ($\deg(d) = \alpha$) if $d \neq 0$ and $d(\mathcal{B}(q)_\beta) \subset \mathcal{B}(q)_{\alpha+\beta}$ for any $\beta \in \mathbb{Z}$. Note that

$$H^1(\mathcal{B}(q)) = \text{Der } \mathcal{B}(q) / \text{ad } \mathcal{B}(q)$$

is the *first cohomology group of $\mathcal{B}(q)$ with coefficients in its adjoint module*.

To begin with, we introduce the following notations ($\alpha \in \mathbb{Z}, i \in \mathbb{Z}_+$)

$$\begin{aligned} \mathcal{B}(q)^{[j]} &= \sum_{\alpha \in \mathbb{Z}} \mathcal{B}(q)_\alpha^{[j]} \quad \text{with} \quad \mathcal{B}(q)_\alpha^{[j]} = \text{span}\{L_{\alpha,i} \mid i \leq j\}, \\ \mathcal{B}(q)^{(j)} &= \sum_{\alpha \in \mathbb{Z}} \mathcal{B}(q)_\alpha^{(j)} \quad \text{with} \quad \mathcal{B}(q)_\alpha^{(j)} = \text{span}\{L_{\alpha,i} \mid i < j\}, \\ \mathcal{B}(q)_\alpha &= \sum_{j \in \mathbb{Z}_+} \mathcal{B}(q)_\alpha^{[j]}, \quad (\text{Der } \mathcal{B}(q))_\alpha = \{d \in \text{Der } \mathcal{B}(q) \mid \deg(d) = \alpha\}. \end{aligned}$$

Clearly, we have a derivation D_0 with degree zero of $\mathcal{B}(q)$ defined by

$$D_0 : L_{\beta,j} \mapsto jL_{\beta,j} \quad \text{for} \quad \beta \in \mathbb{Z}, j \in \mathbb{Z}_+, \quad (3.1)$$

which can be easily checked to be an outer derivation. The main results of this section is given as follows.

Theorem 3.1 *We have $\text{Der } \mathcal{B}(q) = \text{ad } \mathcal{B}(q) \oplus \mathcal{D}$, where $\mathcal{D} = \mathbb{C}D_0$. In particular, the first cohomology group of $\mathcal{B}(q)$ is one-dimensional, namely $\dim H^1(\mathcal{B}(q)) = 1$.*

Proof. Let $d \in \text{Der } \mathcal{B}(q)$. The proof of the theorem is equivalent to proving that d is spanned by an inner derivation $\text{ad}_u \in \text{ad } \mathcal{B}(q)$ for some $u \in \mathcal{B}(q)$ and $D_0 \in \mathcal{D}$. This will be done by Lemmas 3.2–3.6. \square

For a fixed integer $\alpha \in \mathbb{Z}$, consider a nonzero derivation $d \in (\text{Der } \mathcal{B}(q))_\alpha$ such that

$$d(\mathcal{B}(q)^{[j]}) \subset \mathcal{B}(q)^{[i+j]} \quad \text{for any} \quad j \in \mathbb{Z}_+, \quad (3.2)$$

where $i \in \mathbb{Z}$ is assumed to be the minimal integer satisfying (3.2). Then we can write

$$d(L_{\beta,j}) \equiv e_{\beta,j} L_{\alpha+\beta,i+j} \pmod{\mathcal{B}(q)^{(i+j)}}, \quad (3.3)$$

where $e_{\beta,j} \in \mathbb{C}$ and we adopt the convention that if a notation is not defined but technically appears in an expression, we always treat it as zero; for example, if $i < 0$ in (3.2), then $e_{\beta,0} = 0$ for any $\beta \in \mathbb{Z}$.

Lemma 3.2 *The minimal integer i satisfying (3.2) must be nonnegative.*

Proof. If not so, then $i < 0$. Then (3.3) in particular implies $d(L_{\gamma,0}) = 0$ for all $\gamma \in \mathbb{Z}$. Applying d to $[L_{\beta,j}, L_{\gamma,0}] = (\gamma(j+q) - \beta q)L_{\beta+\gamma,j}$, we obtain

$$(\gamma(i+j+q) - (\alpha + \beta)q)e_{\beta,j} = (\gamma(j+q) - \beta q)e_{\beta+\gamma,j}. \quad (3.4)$$

Taking $\gamma = 0$ gives $\alpha e_{\beta,j} = 0$. Using this and replacing (β, γ) by $(\beta, 1)$ and $(\beta + 1, -1)$ respectively in (3.4), we obtain two equations on $e_{\beta,j}$ and $e_{\beta+1,j}$. Solving these two equations, we get

$$(i + 2j + 3q)e_{\beta,j} = 0. \quad (3.5)$$

Furthermore, replacing (β, γ) by $(\beta, 1)$, $(\beta + 1, 1)$ and $(\beta, 2)$ respectively in (3.4), we obtain three equations on $e_{\beta,j}$, $e_{\beta+1,j}$ and $e_{\beta+2,j}$. Solving the three equations, together with (3.5), gives $2(j+q)(j+2q)(2j+3q)e_{\beta,j} = 0$, which implies that $e_{\beta,j} = 0$ for all $\beta \in \mathbb{Z}, j \in \mathbb{Z}_+$, contradicting the minimality of i in (3.2). \square

Lemma 3.3 *If $\alpha \neq 0$ or $\alpha = 0, i \neq 0$, then d in (3.2) is an inner derivation.*

Proof. Applying d to $[L_{\beta,j}, L_{0,0}] = -\beta q L_{\beta,j}$, we have

$$\alpha q e_{\beta,j} = (\alpha(j+q) - \beta(i+q))e_{0,0}. \quad (3.6)$$

First suppose $\alpha \neq 0$. Set $u_1 = (\alpha q)^{-1}e_{0,0}L_{\alpha,i} \in \mathcal{B}(q)$ and let $d' = d + \text{ad}_{u_1}$. It follows that $d'(L_{\beta,j}) \in \mathcal{B}(q)^{(i+j)}$ by (3.6). By induction on i , it follows that d' is an inner derivation. In turn, $d = d' - \text{ad}_{u_1}$ is also an inner derivation.

For the other case $\alpha = 0, i \neq 0$, we see immediately that $e_{0,0} = 0$ by (3.6). Applying d to $[L_{\beta-1,j}, L_{1,0}] = (j + (2 - \beta)q)L_{\beta,j}$ and $[L_{\beta,j}, L_{-1,0}] = -(j + (1 + \beta)q)L_{\beta-1,j}$ respectively, we obtain two equations on $e_{\beta-1,j}$ and $e_{\beta,j}$ as follows

$$\begin{aligned} (i + j + (2 - \beta)q)e_{\beta-1,j} + (j + q + (1 - \beta)(i + q))e_{1,0} &= (j + (2 - \beta)q)e_{\beta,j}, \\ (i + j + (1 + \beta)q)e_{\beta,j} + (j + q + \beta(i + q))e_{-1,0} &= (j + (1 + \beta)q)e_{\beta-1,j}. \end{aligned}$$

In particular, taking $\beta = j = 0$ in the first equation, we see that

$$e_{-1,0} + e_{1,0} = 0. \quad (3.7)$$

In turn, solving the two equations by canceling the term $e_{\beta-1,j}$, together with (3.7), we obtain $i(i + 2j + 3q)(e_{\beta,j} - \beta e_{1,0}) = 0$, which implies that, for $i \neq 0$,

$$e_{\beta,j} = \beta e_{1,0} \text{ for } \beta \in \mathbb{Z}, j \in \mathbb{Z}_+. \quad (3.8)$$

Set $u_2 = (i + q)^{-1}e_{1,0}L_{0,i} \in \mathcal{B}(q)$ and let $d'' = d - \text{ad}_{u_2}$. Then $d''(L_{\beta,j}) \in \mathcal{B}(q)^{(i+j)}$ by (3.8). As in the first case, by induction on i , we know that d'' is an inner derivation, and then d is also an inner derivation. \square

Lemma 3.4 *If $\alpha = i = 0$, then d in (3.2) can be written as $d = \text{ad}_u + \lambda D_0$ for some $u \in \mathcal{B}(q)$ and $\lambda \in \mathbb{C}$.*

Proof. Applying d to $[L_{\beta,j}, L_{\gamma,k}] = (\gamma(j + q) - \beta(k + q)) L_{\beta+\gamma,j+k}$, we have

$$(\gamma(j + q) - \beta(k + q))(e_{\beta,j} + e_{\gamma,k} - e_{\beta+\gamma,j+k}) = 0. \quad (3.9)$$

Replacing (β, γ, k) by $(\beta - 1, 1, 0)$ and $(\beta, -1, 0)$ respectively in (3.9), we have

$$(j + (2 - \beta)q)(e_{\beta-1,j} + e_{1,0} - e_{\beta,j}) = 0, \quad (3.10)$$

$$(j + (1 + \beta)q)(e_{\beta-1,j} - e_{-1,0} - e_{\beta,j}) = 0. \quad (3.11)$$

If $\beta \neq 2 + jq^{-1}$, then $e_{\beta,j} = e_{\beta-1,j} + e_{1,0}$ by (3.10). By induction on β , it follows that

$$e_{\beta,j} = \begin{cases} e_{\beta_0-1,j} + (\beta - \beta_0 + 1)e_{1,0} & \text{if } \beta \leq \beta_0 - 1, \\ e_{\beta_0,j} + (\beta - \beta_0)e_{1,0} & \text{if } \beta \geq \beta_0 + 1, \end{cases} \quad (3.12)$$

where $\beta_0 := [2 + jq^{-1}]$ (the integral part of $2 + jq^{-1}$). If $\beta = 2 + jq^{-1} = \beta_0$, then $e_{\beta_0-1,j} = e_{\beta_0,j} + e_{-1,0} = e_{\beta_0,j} - e_{1,0}$ by (3.11) and (3.7) respectively. This, together with (3.12), gives $e_{\beta,j} = e_{\beta_0,j} + (\beta - \beta_0)e_{1,0}$. In particular, $e_{0,j} = e_{\beta_0,j} - \beta_0 e_{1,0}$, which implies

$$e_{\beta,j} = \beta e_{1,0} + e_{0,j} \quad \text{for } \beta \in \mathbb{Z}, j \in \mathbb{Z}_+. \quad (3.13)$$

Furthermore, substituting (3.13) in (3.9) gives

$$(\gamma(j + q) - \beta(k + q))(e_{0,j} + e_{0,k} - e_{0,j+k}) = 0.$$

Then $e_{0,j+k} = e_{0,j} + e_{0,k}$ by arbitrariness of β or γ . By induction on j , one can derive that $e_{0,j} = je_{0,1}$, which, together with (3.13), gives

$$e_{\beta,j} = \beta e_{1,0} + je_{0,1} \quad \text{for } \beta \in \mathbb{Z}, j \in \mathbb{Z}_+. \quad (3.14)$$

Set $u_3 = q^{-1}e_{1,0}L_{0,0} \in \mathcal{B}(q)$ and let $\bar{d} = d - \text{ad}_{u_3} - e_{0,1}D_0$, where D_0 is defined by (3.1). One will see that $\bar{d}(L_{\beta,j}) \in \mathcal{B}(q)^{(j)}$ by (3.14). By Lemma 3.3, \bar{d} is an inner derivation, and then $d = \text{ad}_u + e_{0,1}D_0$ for some $u \in \mathcal{B}(q)$. This completes the proof. \square

Lemma 3.5 *For every derivation d of degree α , there exists some $i \in \mathbb{Z}_+$ such that (3.2) holds.*

Proof. Obviously, we can choose some $i \in \mathbb{Z}_+$ such that $d(L_{\alpha,0}) \in \mathcal{B}(q)^{[i]}$, $d(L_{0,1}) \in \mathcal{B}(q)^{[i+1]}$ for $\alpha = \pm 1, \pm 2$. Since $\mathcal{B}(q)$ is generated by $\{L_{\alpha,0}, L_{0,1} \mid \alpha = \pm 1, \pm 2\}$, and a derivation is uniquely determined by its action on the generators, we see that (3.2) holds by induction on j . \square

Lemma 3.6 *For every derivation d , there exist derivations d_α of degree α for all $\alpha \in \mathbb{Z}$ such that $d = \sum_{\alpha \in \mathbb{Z}} d_\alpha$, and furthermore, $d_\alpha = 0$ for all but a finite number of α 's.*

Proof. For any $\alpha \in \mathbb{Z}$, we define d_α as follows: Let $x \in \mathcal{B}(q)_\beta$ be any homogenous element of degree $\beta \in \mathbb{Z}$. Suppose $d(x) = \sum_{\gamma \in \mathbb{Z}} y_\gamma$ with $y_\gamma \in \mathcal{B}(q)_\gamma$. Then we set $d_\alpha(x) = y_{\alpha+\beta}$. This uniquely defines a linear map d_α which can be easily verified to be a derivation of degree α . From the definition, we have

$$d = \sum_{\alpha \in \mathbb{Z}} d_\alpha, \quad (3.15)$$

which holds in the sense that for any $x \in \mathcal{B}(q)$, we have $d_\alpha(x) = 0$ for all but a finite many of α 's, and $d(x) = \sum_{\alpha \in \mathbb{Z}} d_\alpha(x)$ (such a sum in (3.15) is *summable* in this sense). This above three lemmas show that exist $u_\alpha \in \mathcal{B}(q)_\alpha$ and some $\lambda_0 \in \mathbb{C}$ such that $d_\alpha = \text{ad}_{u_\alpha} + \delta_{\alpha,0} \lambda_0 D_0$.

Applying (3.15) to $L_{0,0}$, by (1.1), we obtain that $d(L_{0,0}) = -\sum_{\alpha \in \mathbb{Z}} \alpha q u_\alpha$, which in particular implies that $u_\alpha = 0$ for all but a finite number of α 's. \square

4. Central extensions of $\mathcal{B}(q)$

Central extensions play an important role in the theory of Lie algebras, since one can construct many infinite dimensional Lie algebras by central extension and further describe the structures or representations of these Lie algebras. The theory of universal central extensions of Lie algebras over fields is mainly due to Garland [5], in which he constructs a model by using 2-cocycles. On the other hand, the cohomology groups are closely related to the structures of Lie algebras, hence the computation of cohomology groups seems to be important as well.

A 2-cocycle on $\mathcal{B}(q)$ is a \mathbb{C} -bilinear form $\psi : \mathcal{B}(q) \times \mathcal{B}(q) \rightarrow \mathbb{C}$ satisfying the following conditions:

- (i) $\psi(x, y) = -\psi(y, x),$
- (ii) $\psi([x, y], z) + \psi([y, z], x) + \psi([z, x], y) = 0$

for $x, y, z \in \mathcal{B}(q)$. The set of all 2-cocycles on $\mathcal{B}(q)$ is a vector space, denoted by $Z^2(\mathcal{B}(q), \mathbb{C})$. For any \mathbb{C} -linear functions f from $\mathcal{B}(q)$ to \mathbb{C} , define a 2-cocycle ψ_f as follows

$$\psi_f(x, y) = f([x, y]), \quad (4.1)$$

for $x, y \in \mathcal{B}(q)$, which is usually called a *2-coboundary*, or a *trivial 2-cocycle* on $\mathcal{B}(q)$. The set of all 2-coboundaries is a subspace of $Z^2(\mathcal{B}(q), \mathbb{C})$, denoted by $B^2(\mathcal{B}(q), \mathbb{C})$. We say that two 2-cocycles ϕ, ψ are *equivalent* if $\phi - \psi$ is trivial. The quotient space

$$H^2(\mathcal{B}(q), \mathbb{C}) = Z^2(\mathcal{B}(q), \mathbb{C}) / B^2(\mathcal{B}(q), \mathbb{C})$$

is called the *second cohomology group of $\mathcal{B}(q)$ with coefficients in \mathbb{C}* .

As pointed in [2, 6], Vir has the unique nontrivial one-dimensional central extension, namely $\dim H^2(\text{Vir}, \mathbb{C}) = 1$. In this section, we shall determine the central extension of $\mathcal{B}(q)$ and the second cohomology group $H^2(\mathcal{B}(q), \mathbb{C})$. In fact, we obtain the following analogous results of $\mathcal{B}(q)$.

Theorem 4.1 *The unique nontrivial central extension of $\mathcal{B}(q)$ is given by*

$$[L_{\alpha,i}, L_{\beta,j}] = (\beta(i+q) - \alpha(j+q)) L_{\alpha+\beta,i+j} + \phi(L_{\alpha,i}, L_{\beta,j})c \quad (4.2)$$

for $\alpha, \beta \in \mathbb{Z}$, $i, j \in \mathbb{Z}_+$, where c is a central element and ϕ is the following non-trivial 2-cocycle

$$\phi(L_{\alpha,i}, L_{\beta,j}) = \delta_{\alpha+\beta,0} \delta_{i,0} \delta_{j,0} \frac{\alpha^3 - \alpha}{12}. \quad (4.3)$$

Hence the second cohomology group of $\mathcal{B}(q)$ is $H^2(\mathcal{B}(q), \mathbb{C}) = \mathbb{C}\phi$.

Proof. We shall prove the above theorem in a series of lemmas. In particular, Lemmas 4.3 and 4.4 imply that a 2-cocycle ϕ to be defined later takes the required form (4.3). This non-trivial 2-cocycle ϕ induces the central extension of $\mathcal{B}(q)$ as (4.2) by taking $c = 2\phi(L_{2,0}, L_{-2,0})$ (see Lemma 4.4(2)). \square

First, we have

$$[L_{\alpha,i}, L_{0,0}] = -\alpha q L_{\alpha,i}, \quad [L_{-1,i}, L_{1,0}] = (i+2q) L_{0,i}.$$

Let ψ be any 2-cocycle. Define a linear function on $\mathcal{B}(q)$ as follows:

$$f(L_{\alpha,i}) = \begin{cases} -(\alpha q)^{-1} \psi(L_{\alpha,i}, L_{0,0}) & \text{if } \alpha \neq 0, \\ (i+2q)^{-1} \psi(L_{-1,i}, L_{1,0}) & \text{otherwise.} \end{cases} \quad (4.4)$$

Then $\phi = \psi - \psi_f$ is a 2-cocycle of $\mathcal{B}(q)$, which is equivalent to ψ , where ψ_f is the trivial 2-cocycle induced by f as in (4.1). Thus, by (4.4), we immediately have

$$\phi(L_{\alpha,i}, L_{0,0}) = 0 \quad \text{if } \alpha \neq 0, \quad (4.5)$$

$$\phi(L_{-1,i}, L_{1,0}) = 0. \quad (4.6)$$

Lemma 4.2 *If $i \neq 0$, then $\phi(L_{-2,i}, L_{2,0}) = 0$.*

Proof. Applying ϕ to triple $(L_{0,0}, L_{1,0}, L_{-1,i})$, by (4.6), we have

$$\begin{aligned} 0 &= \frac{1}{i+2q} (\phi([L_{-1,i}, L_{0,0}], L_{1,0}) + \phi([L_{0,0}, L_{1,0}], L_{-1,i})) \\ &= \frac{1}{i+2q} \phi(L_{0,0}, [L_{1,0}, L_{-1,i}]) = \phi(L_{0,i}, L_{0,0}). \end{aligned}$$

In turn, applying ϕ to triple $(L_{0,i}, L_{1,0}, L_{-1,0})$, it follows from the above formula and (4.6) that

$$\begin{aligned} 0 &= \frac{1}{i+q} \phi(L_{0,i}, [L_{1,0}, L_{-1,0}]) \\ &= \frac{1}{i+q} (\phi([L_{0,i}, L_{1,0}], L_{-1,0}) + \phi([L_{-1,0}, L_{0,i}], L_{1,0})) \\ &= \phi(L_{1,i}, L_{-1,0}). \end{aligned} \quad (4.7)$$

So, applying ϕ to triple $(L_{2,0}, L_{-1,i}, L_{-1,0})$, by (4.6) and (4.7), we have

$$\begin{aligned} 0 &= \phi([L_{-1,0}, L_{2,0}], L_{-1,i}) + \phi([L_{2,0}, L_{-1,i}], L_{-1,0}) \\ &= \phi(L_{2,0}, [L_{-1,i}, L_{-1,0}]) = i\phi(L_{-2,i}, L_{2,0}), \end{aligned}$$

which gives the desired conclusion. \square

Lemma 4.3 *For $\alpha, \beta \in \mathbb{Z}$, $i, j \in \mathbb{Z}_+$, we have*

- (1) $\phi(L_{\alpha,i}, L_{\beta,j}) = 0$ if $\alpha + \beta \neq 0$;
- (2) $\phi(L_{\alpha,i}, L_{-\alpha,j}) = 0$ if $i \neq j$.

Proof. If $\alpha + \beta \neq 0$, applying ϕ to triple $(L_{0,0}, L_{\alpha,i}, L_{\beta,j})$, by (4.5), we see

$$\begin{aligned} 0 &= \frac{1}{(\alpha+\beta)q} \phi(L_{0,0}, [L_{\alpha,i}, L_{\beta,j}]) \\ &= \frac{1}{(\alpha+\beta)q} (\phi([L_{0,0}, L_{\alpha,i}], L_{\beta,j}) + \phi([L_{\beta,j}, L_{0,0}], L_{\alpha,i})) \\ &= \phi(L_{\alpha,i}, L_{\beta,j}), \end{aligned}$$

which gives part (1). Applying ϕ to triples $(L_{1,0}, L_{\alpha,i}, L_{-1-\alpha,j})$ and $(L_{-1,0}, L_{1+\alpha,i}, L_{-\alpha,j})$ respectively, by (4.6) and (4.7), we obtain following two equations

$$\begin{aligned}
0 &= \phi(L_{1,0}, [L_{\alpha,i}, L_{-1-\alpha,j}]) \\
&= \phi([L_{1,0}, L_{\alpha,i}], L_{-1-\alpha,j}) + \phi([L_{-1-\alpha,j}, L_{1,0}], L_{\alpha,i}) \\
&= ((\alpha - 1)q - i)\phi(L_{1+\alpha,i}, L_{-1-\alpha,j}) - ((\alpha + 2)q + j)\phi(L_{\alpha,i}, L_{-\alpha,j}), \\
0 &= \phi(L_{-1,0}, [L_{1+\alpha,i}, L_{-\alpha,j}]) \\
&= \phi([L_{-1,0}, L_{1+\alpha,i}], L_{-\alpha,j}) + \phi([L_{-\alpha,j}, L_{-1,0}], L_{1+\alpha,i}) \\
&= ((\alpha + 2)q + i)\phi(L_{\alpha,i}, L_{-\alpha,j}) - ((\alpha - 1)q - j)\phi(L_{1+\alpha,i}, L_{-1-\alpha,j}).
\end{aligned}$$

Multiplying the first equation by $(\alpha - 1)q - j$, the second one by $(\alpha - 1)q - i$, and then adding both results together, we deduce

$$(i - j)(i + j + 3q)\phi(L_{\alpha,i}, L_{-\alpha,j}) = 0,$$

which immediately gives part (2) since $i, j \in \mathbb{Z}_+$ and $q > 0$. □

Lemma 4.4 *For $\alpha \in \mathbb{Z}$, $i \in \mathbb{Z}_+$, we have*

- (1) $\phi(L_{\alpha,i}, L_{-\alpha,i}) = 0$ if $i \neq 0$;
- (2) $\phi(L_{\alpha,0}, L_{-\alpha,0}) = \frac{\alpha^3 - \alpha}{6}\phi(L_{2,0}, L_{-2,0})$.

Proof. Applying ϕ to triple $(L_{2,0}, L_{-2-\alpha,i}, L_{\alpha,i})$, we get

$$\begin{aligned}
2(i + q)(\alpha + 1)\phi(L_{2,0}, L_{-2,2i}) &= \phi(L_{2,0}, [L_{-2-\alpha,i}, L_{\alpha,i}]) \\
&= \phi([L_{2,0}, L_{-2-\alpha,i}], L_{\alpha,i}) + \phi([L_{\alpha,i}, L_{2,0}], L_{-2-\alpha,i}) \\
&= ((\alpha + 4)q + 2i)\phi(L_{\alpha,i}, L_{-\alpha,i}) - ((\alpha - 2)q - 2i)\phi(L_{2+\alpha,i}, L_{-2-\alpha,i}).
\end{aligned} \tag{4.8}$$

Furthermore, applying ϕ to triples $(L_{1,0}, L_{\alpha,i}, L_{-1-\alpha,i})$ and $(L_{1,0}, L_{1+\alpha,i}, L_{-2-\alpha,i})$ respectively (here we omit the details), by (4.6), we have

$$((\alpha + 2)q + i)\phi(L_{\alpha,i}, L_{-\alpha,i}) = ((\alpha - 1)q - i)\phi(L_{1+\alpha,i}, L_{-1-\alpha,i}), \tag{4.9}$$

$$((\alpha + 3)q + i)\phi(L_{1+\alpha,i}, L_{-1-\alpha,i}) = (\alpha q - i)\phi(L_{2+\alpha,i}, L_{-2-\alpha,i}). \tag{4.10}$$

Solving the three equations (4.8)–(4.10) by canceling the common terms $\phi(L_{1+\alpha,i}, L_{-1-\alpha,i})$ and $\phi(L_{2+\alpha,i}, L_{-2-\alpha,i})$, we obtain

$$(i + 2q)(2i + 3q)\phi(L_{\alpha,i}, L_{-\alpha,i}) = (\alpha + 1)(\alpha q - i)((\alpha - 1)q - i)\phi(L_{2,0}, L_{-2,2i}).$$

If $i \neq 0$, then the above formula gives part (1) by Lemma 4.2. Otherwise, $i = 0$ and part (2) clearly holds. \square

Acknowledgement The authors would like to thank Professor Yucai Su for instructions and helps.

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